

## INTRODUCTION TO MATRIX ARITHMETIC

A *matrix* is a rectangular array of numbers. The *dimensions* of the matrix are the number of rows and columns. An *r-by-c matrix* has *r* rows and *c* columns.

$$\begin{bmatrix} 8 & 9 \\ 1 & 4 \end{bmatrix}$$

a 2-by-2 matrix

$$\begin{bmatrix} 42 \\ 61 \\ -7 \\ 846 \end{bmatrix}$$

a 4-by-1 matrix

$$\begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \\ 2 & 6 & 5 \\ 3 & 5 & 8 \end{bmatrix}$$

a 4-by-3 matrix

We can do the basic arithmetic operations (addition, subtraction, multiplication, division) on matrices, as we can with regular numbers. It's "just the same ... only different."

### ADDITION AND SUBTRACTION:

Two matrices of the same size (same number of rows, and same number of columns) can be added or subtracted on a term-by-term basis.

For example:

$$\begin{bmatrix} 8 & 2 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 3 \\ 7 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 2 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ -1 & 2 \end{bmatrix}$$

### MULTIPLICATION:

Matrix multiplication is a tad more involved. The basic algorithm is "across the row and down the column." That is, working across the row, you multiply each element in the first matrix by a corresponding element (working down the column) in the second matrix, and then add these products. An example is far easier to understand than a description.

Let's multiply the following two matrices:

$$\begin{bmatrix} 2 & 1 & 3 \\ 2 & 5 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 7 \\ 3 & 2 \\ 6 & 3 \end{bmatrix}$$

Notice that the two matrices are NOT the same size. The first is 2-by-3, the second is 3-by-2. In order to multiply two matrices, we need the number of columns in the first matrix to equal the number of rows in the second matrix.

Let's go "across the row and down the column" for the first row of Matrix 1 and the first column of Matrix 2:

$$2*1 + 1*3 + 3*6 = 23$$

This is the row-1-and-column-1 number in the resulting product.

The rest of the matrix is as follows:

$$\text{Row 1 and Column 2} = 2*7 + 1*2 + 3*3 = 25$$

$$\text{Row 2 and Column 1} = 2*1 + 5*3 + 4*6 = 41$$

$$\text{Row 2 and Column 2} = 2*7 + 5*2 + 4*3 = 36$$

And so the resulting product of the two matrices is

$$\begin{bmatrix} 2 & 1 & 3 \\ 2 & 5 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 7 \\ 3 & 2 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} 23 & 25 \\ 41 & 36 \end{bmatrix}.$$

In regular arithmetic, it doesn't matter what order you multiply things in – 3 times 6 is the same as 6 times 3. This is NOT the case with matrix multiplication. Let's take the same two matrices from last time, but multiply them in reverse order. Thus:

$$\begin{bmatrix} 1 & 7 \\ 3 & 2 \\ 6 & 3 \end{bmatrix} \times \begin{bmatrix} 2 & 1 & 3 \\ 2 & 5 & 4 \end{bmatrix} = \begin{bmatrix} 16 & 36 & 31 \\ 10 & 13 & 17 \\ 18 & 21 & 30 \end{bmatrix}.$$

#### THE IDENTITY MATRIX:

A square matrix (same number of rows and columns) consisting only of 1's down the main diagonal and 0's everywhere else is called the "identity matrix." Multiplying by the identity matrix, in matrix arithmetic, has the same effect as multiplying by 1 in ordinary arithmetic. Thus, for example:

$$\begin{bmatrix} 3 & 2 \\ -1 & 6 \\ 0 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -1 & 6 \\ 0 & 4 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 3 & 2 \\ -1 & 6 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -1 & 6 \\ 0 & 4 \end{bmatrix}.$$

### THE INVERSE OF A MATRIX:

Multiplying a matrix  $\mathbf{A}$  by its inverse  $\mathbf{A}^{-1}$  gives the identity matrix,  $\mathbf{I}$ . Only square matrices have inverses. Thus, for example, the matrix  $\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$  has as its inverse  $\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$ , since

$$\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \times \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The inverse of a 2-by-2 matrix can be found quite easily. In general, the inverse of the matrix  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  is

$$\frac{1}{ad - bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Finding inverses for matrices larger than two-by-two is a more involved process computationally. If this were a course on linear algebra, we'd cover it. Since it's not, we'll just have the computer give us the inverse of larger matrices.

### DETERMINANT OF A MATRIX:

Every square matrix has a number called a *determinant* associated with it. Technically, the determinant of a matrix  $\mathbf{A}$  is the sum of all the signed elementary products of  $\mathbf{A}$ . That definition doesn't make much sense without doing a bit of background definitions and computations – good stuff for a linear algebra course, but not particularly appropriate for this class.

One way of conceptualizing the determinant is to think of it as being (in some abstract sense) a measure of the “size” of the matrix ... not in “number of rows” terms but in “magnitude” terms.

For a two-by-two matrix, the determinant is simple. The determinant of  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  is just  $ad - bc$ . For a three-by-three matrix, things are a bit sloppier, but are still manageable. The determinant of

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

is

$$a e i + b f g + c d h - c e g - a f h - b d i.$$

(Note the pattern in the computations.)

Determinants play several roles in matrix arithmetic. The key point for this class is that if the determinant is 0, then the matrix cannot be inverted. (This is the matrix counterpart of dividing by 0.)